Math 3280 23-09-18

Review

- Sample spaces having equally likely outcomes.

$$P(E) = \frac{\# E}{\# S}.$$

- Conditional prob.

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

(Multiplicative rule) $P(E_1 \cdots E_n)$

$$= P(E_1)\, P(E_2|E_1)\, P(E_3|E_1E_2) \cdots P(E_n|E_1 \cdots E_{n-1})$$

Exer 1. Two fair dice are rolled. What is the conditional probability that at least one lands on 6 given that the dice land on different numbers?

Solution. Let E denote the event that at least one die lands on 6, and let F denote the event that the dice land on different numbers.

Then

$$E = \{ (i,j) \in \{1,2,3,4,5,6\}^2 : i=6 \text{ or } j=6 \}$$

$$F = \{ (i,j) \in \{1,2,3,4,5,6\}^2 : i \neq j \}.$$

$$E \cap F = \{ (1,6), (2,6), \cdots, (5,6), (6,1), (6,2), \cdots, (6,5) \}$$

Notice that $\#(E \cap F) = 10$

$$\# F = 6 \times 5 = 30$$

Hence

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\#(E \cap F) / \# S}{\# F / \# S}$$

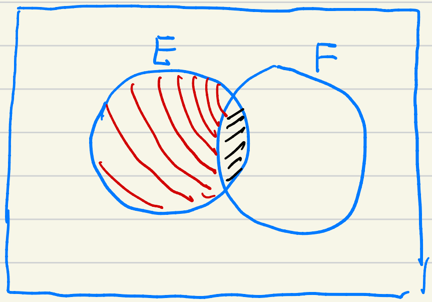
$$= \frac{\#(E \cap F)}{\# F}$$

$$= \frac{10}{30} = \frac{1}{3}.$$

§ 3.2

Bayes' formula.

Let E, F be two events.



$$E = (E \cap F) \cup (E \cap F^c)$$

(black). (red)

Hence

$$P(E) = P(E \cap F) + P(E \cap F^c)$$

But $\quad P(E \cap F) = P(F) \cdot P(E|F),$

$$P(E \cap F^c) = P(F^c) \cdot P(E|F_c).$$

We obtain

$$P(E) = P(F) \cdot P(E|F) + P(F^c) \cdot P(E|F^c).$$

(Total probability formula).

Hence to determine the prob. of E,
we may first conduct the "conditioning"
upon whether or not the event F has
occured

Next we give a generalization of this formula.

Let $F_1, F_2, \cdots, F_n$ be a sequence of events
such that they are mutually exclusive,
and $\bigcup_{k=1}^{n} F_k = S$ (we say $F_1, \cdots, F_n$ are exhaustive)

Then we have

$$P(E) = \sum_{k=1}^{n} P(F_k) \cdot P(E \mid F_k).$$

pf: Notice that $E = \bigcup_{k=1}^{n} (E \cap F_k)$

(with disjoint union)

Hence

$$P(E) = \sum_{k=1}^{n} P(E \cap F_k)$$

$$= \sum_{k=1}^{n} P(F_k) P(E|F_k). \quad \boxed{/\!/\!/}$$

Prop. (Bayes' formula).

Assume $F_1, \dots, F_n$ are mutually exclusive and exhaustive.

Then for any $1 \leq i \leq n$,

$$P(F_i|E) = \frac{P(F_i) \cdot P(E|F_i)}{\sum_{k=1}^{n} P(F_k) P(E|F_k)}$$

pf: $\sum_{k=1}^{n} P(F_k) P(E|F_k) = P(E)$

$$P(F_i) \cdot P(E|F_i) = P(EF_i)$$

$\boxed{/\!/\!/}$

Example 3.

A bin contains 3 types of disposable flashlights. The probability that a type 1 flashlight will give more than 100 hours of use is .7, with the corresponding probabilities for type 2 and type 3 flashlights being .4 and .3, respectively. Suppose that 20 percent of the flashlights in the bin are type 1, 30 percent are type 2, and 50 percent are type 3.

(a) What is the probability that a randomly chosen flashlight will give more than 100 hours of use?
(b) Given that a flashlight lasted more than 100 hours, what is the conditional probability that it was a type j flashlight, j = 1, 2, 3?

Solution: E —— the event that a random chosen flashlight will give more than 100 hours.

F_i $(i=1,2,3)$ —— the event that a random chosen flashlight is of type i.

We need to find out (a) $P(E)$

(b) $P(F_i | E)$.

From the conditions of the question, we know

$$P(E|F_1) = 0.7, \quad P(E|F_2) = 0.4$$

$$P(E|F_3) = 0.3.$$

$$P(F_1) = 0.2, \quad P(F_2) = 0.3, \quad P(F_3) = 0.5.$$

Hence $$P(E) = \sum_{i=1}^{3} P(F_i)\, P(E|F_i)$$

$$= 0.2 \times 0.7 + 0.3 \times 0.4 + 0.5 \times 0.3$$

$$P(F_1|E) = \frac{P(F_1) \cdot P(E|F_1)}{P(E)}$$

$$= \frac{0.2 \times 0.7}{0.2 \times 0.7 + 0.3 \times 0.4 + 0.5 \times 0.3}.$$

$$= \frac{14}{41}$$

Similarly

$$P(F_2|E) = \frac{12}{41}, \quad P(F_3|E) = \frac{15}{41}. \quad \square$$